

REMARKS ON THE COLLAPSING OF TORUS FIBERED CALABI-YAU MANIFOLDS

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ABSTRACT. One of the main results of the paper [6] by Gross-Tosatti-Zhang establishes estimates on the collapsing of Ricci-flat Kähler metrics on holomorphic torus fibrations. We remove a projectivity assumption from these estimates and simplify some of the underlying analysis.

1. INTRODUCTION

Let $f : M \rightarrow N$ be a surjective holomorphic map between compact Kähler manifolds. We assume that M is Calabi-Yau, i.e. $c_1(M)_{\mathbb{R}} = 0$, and that all smooth fibers of f are complex n -tori, i.e. $M_y = f^{-1}(y) = \mathbb{C}^n / \Lambda_y$ for all $y \in N \setminus f(S)$, where S is the set of critical points of f . Fix Kähler metrics ω_M, ω_N on M, N , put $\omega_0 = f^* \omega_N$, and for all $t \in (0, 1]$ let $\tilde{\omega}_t$ be the unique Ricci-flat Kähler metric on M cohomologous to $\omega_0 + t\omega_M$, whose existence is guaranteed by Yau's theorem [14].

The goal of this note is to prove the following estimate for $\tilde{\omega}_t$:

Theorem 1.1. *Given any compact set $K \subset M \setminus f^{-1}(f(S))$ and any $k \in \mathbb{N}_0$, there exists a constant $C_{K,k} < \infty$, which does not depend on t , such that*

$$\|\tilde{\omega}_t\|_{C^k(K, \omega_M)} \leq C_{K,k} \quad (1.1)$$

holds uniformly for all $t \in (0, 1]$.

Theorem 1.1 was proved in [6] (see Proposition 4.6 there) assuming that M is projective algebraic, and our first purpose here is precisely to remove this assumption. Our second purpose is to simplify the analysis in [6] by applying a local Calabi-Yau type C^3 estimate in place of Evans-Krylov theory, so that the somewhat delicate $i\partial\bar{\partial}$ -lemma of [6, Proposition 3.1] is no longer needed.

The proofs of the first two main theorems of [6] (i.e. Theorems 1.1 and 1.2 there) did not rely on the assumption that M is projective except through the proof of (1.1). Thus, we are also removing the projectivity assumption from those two results. Similar remarks apply to [3, Section 5].

Let us now quickly sketch the contents of this note. We fix a small coordinate ball $B \subset N \setminus f(S)$ and call $U = f^{-1}(B)$. Recall that a closed real $(1, 1)$ -form on U is called *semi-flat* if it restricts to a flat Kähler metric on every fiber $f^{-1}(y)$ ($y \in B$). Let $p : B \times \mathbb{C}^n \rightarrow U$ be the universal holomorphic cover of U . The key result to deriving (1.1) is the following:

Theorem 1.2. *There exists a semi-flat form $\omega_{\text{SF}} \geq 0$ on U such that $p^* \omega_{\text{SF}} = i\partial\bar{\partial}\eta$ for a smooth real-valued function η on $B \times \mathbb{C}^n$ with the scaling property*

$$\eta(y, \lambda z) = \lambda^2 \eta(y, z) \quad (\lambda \in \mathbb{R}). \quad (1.2)$$

This was proved in [6, Section 3], under the assumption that M is projective, using the fact that polarized abelian varieties are classified by the Siegel upper half-space. Our observation here, which lets us remove the projectivity condition, is that the Siegel upper half-space more generally classifies (marked) polarized *complex tori*: pairs consisting of a complex n -torus T and a group isomorphism $\phi : \Lambda^2 \mathbb{Z}^{2n} \rightarrow H^2(T, \mathbb{Z})$ such that $\phi_{\mathbb{R}}(x)$ is a Kähler class on T , where $x \in \Lambda^2 \mathbb{R}^{2n}$ is given and fixed

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but not necessarily rational. This is implicit in papers of Fujiki [4, Proposition 14] and Schumacher [10, Theorem 4.4], and we will make it more explicit in the proof of Theorem 1.2 in Section 2.

Section 3 then completes the proof of Theorem 1.1 along the lines indicated above.

2. CONSTRUCTION OF THE SEMI-FLAT FORM

We will show Theorem 1.2 by writing down an explicit formula for η on $B \times \mathbb{C}^n$ and checking that $i\partial\bar{\partial}\eta$ is invariant under the automorphism group of the covering p . In fact, the formula for η takes the following form, from which (1.2) and the semi-flat property are clear:

$$\eta(y, z) = -\frac{1}{4} \sum_{\ell, m=1}^n (\operatorname{Im} Z(y))_{\ell m}^{-1} (z_\ell - \bar{z}_\ell)(z_m - \bar{z}_m). \quad (2.1)$$

Here Z denotes an appropriately constructed *period map* from B to the Siegel upper half-space \mathfrak{H}_n of symmetric $n \times n$ complex matrices with positive definite imaginary parts. It remains to explain the construction of Z , which will be done in two steps and represents the main difference between this section and [6, Section 3], and to check translation invariance and semi-positivity of $i\partial\bar{\partial}\eta$.

2.1. Construction of a polarization. Fix a basis $(v_1(y), \dots, v_{2n}(y))$ of the lattice Λ_y that varies holomorphically with $y \in B$, and let $(\xi^1(y), \dots, \xi^{2n}(y))$ be the \mathbb{R} -dual basis of 1-forms on \mathbb{C}^n . Then the classes $[\xi^i(y) \wedge \xi^j(y)] \in H^2(M_y, \mathbb{Z}) \subset H^2(M_y, \mathbb{R})$ with $i < j$ form a basis of $H^2(M_y, \mathbb{Z})$.

Proposition 2.1. *Let Ω be a real 2-form on U whose restriction to M_y is closed for all $y \in B$, and expand $[\Omega|_{M_y}] = \sum_{i < j} P_{ij}(y)[\xi^i(y) \wedge \xi^j(y)]$. If Ω is closed, then the $P_{ij}(y)$ do not depend on y .*

Proof. We use the Gauss-Manin connection ∇^{GM} on the smooth \mathbb{R} -vector bundle $R^2 f_* \mathbb{R} \otimes \mathcal{C}_B^\infty$. By definition, the sections $[\xi^i(y) \wedge \xi^j(y)]$ of this vector bundle form a basis of the space of ∇^{GM} -parallel sections. On the other hand, since Ω is closed, Cartan's magic formula yields that $\nabla^{\text{GM}}[\Omega|_{M_y}] = 0$; see [2, Corollary 4.4.4] or [13, Proposition 9.14]. This immediately implies the claim. \square

We apply this to the Kähler form $\Omega = \omega_M$. The only difference with the projective case considered in [6] is that there, we could have chosen the $P_{ij}(y)$ to be \mathbb{Z} -valued to begin with and hence *trivially* independent of y . Now let $Q \in \mathbb{R}^{2n \times 2n}$ denote the unique skew-symmetric matrix with $Q_{ij} = P_{ij}$ for all $i < j$ and define, for each $y \in B$, a closed real 2-form ω_y on the torus M_y by setting

$$\omega_y = \frac{1}{2} \sum Q_{ij} \xi^i(y) \wedge \xi^j(y). \quad (2.2)$$

Notice that ω_y is translation-invariant and cohomologous to $\omega_M|_{M_y}$. Thus ω_y is itself a Kähler form and indeed the unique flat Kähler form on M_y cohomologous to $\omega_M|_{M_y}$.

Remark 2.2. An interesting class of complex n -torus bundles whose total spaces do not admit *any* Kähler forms was introduced by Atiyah [1]. The base of all of these bundles is the projective variety $\text{SO}(2n)/\text{U}(n)$ parametrizing linear complex structures on \mathbb{C}^n compatible with the Euclidean metric and standard orientation. For $n = 2$ they are precisely the twistor spaces of abelian surfaces.

In fact, any choice of a polarization $x \in \Lambda^2 \mathbb{R}^{2n}$ as at the end of Section 1 defines an embedding F_x of the Stein manifold \mathfrak{H}_n into the moduli space $\mathfrak{M}_n = \text{GL}(2n, \mathbb{R})/\text{GL}(n, \mathbb{C})$ of all complex n -tori, and for each $T \in F_x(\mathfrak{H}_n)$ there exists a natural subvariety $X_{T,x} \subset \mathfrak{M}_n$, isomorphic to $\text{SO}(2n)/\text{U}(n)$ and intersecting $F_x(\mathfrak{H}_n)$ transversely in T , which serves as the base of an Atiyah bundle.

2.2. Construction of the period map. From here on, we can follow the arguments in [8, p. 371]¹ to construct a holomorphic period map $Z : B \rightarrow \mathfrak{H}_n$. We will give some more details than in [8].

We begin by setting $T = (v_1, \dots, v_{2n}) \in \mathcal{O}(B, \mathbb{C}^{n \times 2n})$. Also, we choose $S \in \text{GL}(2n, \mathbb{R})$ such that $S^{\text{tr}} Q S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ in terms of the canonical decomposition $\mathbb{R}^{2n} = \mathbb{R}^n \oplus \mathbb{R}^n$ (here and in the following we denote by A^{tr} the transpose of a square matrix A). Observe that S is unique only up to right

¹The discussion in [8] involves a certain 2-form ω on U (not assumed to be $(1, 1)$ or closed) but in fact only depends on the family of fiberwise 2-forms $\{\omega|_{M_y}\}_{y \in B}$. In our case this family is defined by (2.2).

multiplication by $\mathrm{Sp}(2n, \mathbb{R})$ and that the period map Z to be defined momentarily will depend on this choice, but only up to the action of a holomorphic isometry of \mathfrak{H}_n . In the course of the proof of Proposition 2.4, we will see that the first n columns of TS are \mathbb{C} -linearly independent. We can therefore write $TS = R(1, Z)$ with $R \in \mathcal{O}(B, \mathrm{GL}(n, \mathbb{C}))$ and $Z \in \mathcal{O}(B, \mathbb{C}^{n \times n})$.

Remark 2.3. We have an additional freedom of choosing a lattice basis, i.e. of replacing T by TA for some constant matrix $A \in \mathrm{GL}(2n, \mathbb{Z})$. Then Q changes to $A^{\mathrm{tr}}QA$, but S becomes $A^{-1}S$ so that the period map Z remains unchanged. We will not make use of this freedom here, but if Q is integral as in [6], then we could arrange in this way that $Q = \begin{pmatrix} 0 & \Delta \\ -\Delta & 0 \end{pmatrix}$, where $\Delta = \mathrm{diag}(d_1, \dots, d_n)$ for some positive integers $d_1|d_2|\dots|d_n$; compare [5, p. 304, Lemma].

We now work at a given point $y \in B$ and for simplicity write $\omega = \omega_y$ and $Z = Z(y)$. Recall that, even though this was not immediate from (2.2), $\omega = \omega_y$ is indeed a Kähler form on M_y .

Proposition 2.4. (a) *It holds that $Z \in \mathfrak{H}_n$. In fact, this is equivalent to ω being positive $(1, 1)$.*

(b) *Moreover, $\omega = i \sum H_{\ell m} dz_\ell \wedge d\bar{z}_m$, where $H^{-1} = 2\bar{R}(\mathrm{Im} Z)R^{\mathrm{tr}} = i\bar{T}Q^{-1}T^{\mathrm{tr}}$.*

Proof. We view the columns of TS as an \mathbb{R} -basis of \mathbb{C}^n and denote the \mathbb{R} -dual basis by $\zeta^1, \dots, \zeta^{2n}$. Then $\xi^i = \sum S_{ij}\zeta^j$ and hence $\omega = \sum_{k=1}^n \zeta^k \wedge \zeta^{n+k}$. As an aside, we can now deduce that the first n columns of TS are \mathbb{C} -linearly independent, thus justifying the construction of R and Z . Indeed, let V denote their \mathbb{R} -span in \mathbb{C}^n . Then $\omega(v, iv) = 0$ for all $v \in V \cap iV$ by the preceding formula; since ω is positive $(1, 1)$, this means that the \mathbb{C} -subspace $V \cap iV$ is trivial, as desired.

The matrix R defines a \mathbb{C} -isomorphism $R : \mathbb{C}^n \rightarrow \mathbb{C}^n$, so ω is positive $(1, 1)$ if and only if $R^*\omega$ is. But $R^*\omega = \sum_{k=1}^n \eta^k \wedge \eta^{n+k}$ for the basis of 1-forms η^1, \dots, η^{2n} that is \mathbb{R} -dual to the basis of column vectors of $(1, Z)$. To understand the condition for this form to be positive $(1, 1)$, we define matrices $A, B, \tilde{A}, \tilde{B} \in \mathbb{C}^{n \times n}$ by $\eta^k = \sum A_{k\ell} dz_\ell + B_{k\ell} d\bar{z}_\ell$ and $\eta^{n+k} = \sum \tilde{A}_{k\ell} dz_\ell + \tilde{B}_{k\ell} d\bar{z}_\ell$ for $k, \ell \in \{1, \dots, n\}$. Denoting the skew-symmetric part of a matrix by a superscript “skew”, we then have that

$$R^*\omega = \sum (A^{\mathrm{tr}}\tilde{A})_{\ell m}^{\mathrm{skew}} dz_\ell \wedge dz_m + (A^{\mathrm{tr}}\tilde{B} - \tilde{A}^{\mathrm{tr}}B)_{\ell m} dz_\ell \wedge d\bar{z}_m + (B^{\mathrm{tr}}\tilde{B})_{\ell m}^{\mathrm{skew}} d\bar{z}_\ell \wedge d\bar{z}_m,$$

so $R^*\omega$ is positive $(1, 1)$ if and only if $(A^{\mathrm{tr}}\tilde{A})^{\mathrm{skew}} = (B^{\mathrm{tr}}\tilde{B})^{\mathrm{skew}} = 0$ and $\frac{1}{i}(A^{\mathrm{tr}}\tilde{B} - \tilde{A}^{\mathrm{tr}}B)$ is Hermitian positive definite. In order to rephrase this in terms of Z , observe that we have

$$A + B = 1, \quad AZ + B\bar{Z} = 0, \quad \tilde{A} + \tilde{B} = 0, \quad \tilde{A}Z + \tilde{B}\bar{Z} = 1.$$

This yields that $\tilde{A}(Z - \bar{Z}) = 1$ (so, in particular, both factors are invertible) and $A = -\bar{Z}\tilde{A}$. Given this, $(A^{\mathrm{tr}}\tilde{A})^{\mathrm{skew}} = 0$ is clearly equivalent to Z being symmetric, and then $\frac{1}{i}(A^{\mathrm{tr}}\tilde{B} - \tilde{A}^{\mathrm{tr}}B) = i\tilde{A}^{\mathrm{tr}} = \frac{1}{2}(\mathrm{Im} Z)^{-1} > 0$. Thus, $Z \in \mathfrak{H}_n$. Also, $R^*\omega = \frac{i}{2} \sum (\mathrm{Im} Z)_{\ell m}^{-1} dz_\ell \wedge d\bar{z}_m$, which implies (b). \square

2.3. Comparison with [6]. Let us assume for the moment that Q is integral as in [6]. We wish to compare our formalism, specialized to this case, with the treatment in [6].

We can assume without loss that Q is as in Remark 2.3. Also, let us write $\Delta = \Sigma^2$, where $\Sigma > 0$. Then we choose $S = \mathrm{diag}(\Sigma^{-1}, \Sigma^{-1})$, so that $T = (R\Sigma, RZ\Sigma)$. Finally, we apply the automorphism R^{-1} to simplify the picture, mapping T to $(\Sigma, Z\Sigma)$ and ω to $\frac{i}{2} \sum (\mathrm{Im} Z)_{\ell m}^{-1} dz_\ell \wedge d\bar{z}_m$.

On the other hand, [6, p. 528] states that we can assume that $T = (\Delta, Z_0)$ with $Z_0 \in \mathfrak{H}_n$ and that $\omega = \frac{i}{2} \sum (\mathrm{Im} Z_0)_{\ell m}^{-1} dz_\ell \wedge d\bar{z}_m$. More precisely, this means that, given the original $T = (v_1, \dots, v_{2n})$, there exists $R_0 \in \mathrm{GL}(n, \mathbb{C})$ such that $T = R_0(\Delta, Z_0)$ and such that $R_0^*\omega$ equals the above. We then recover the previous picture by setting $R = R_0\Sigma$ and $Z = \Sigma^{-1}Z_0\Sigma^{-1}$.

2.4. Translation invariance and semi-positivity. We now return to the general case. Again we apply R^{-1} to simplify matters, so that now $T = (1, Z)S^{-1}$ and $\omega = \frac{i}{2} \sum (\mathrm{Im} Z)_{\ell m}^{-1} dz_\ell \wedge d\bar{z}_m$.

It remains to check that, for η as in (2.1), the form $i\partial\bar{\partial}\eta$ is nonnegative and invariant under the deck transformations of the covering $p : B \times \mathbb{C}^n \rightarrow U$. (Of course the resulting semi-flat form on U will then satisfy $\omega_{\mathrm{SF}}|_{M_y} = \omega_y$ for all $y \in B$, but we do not need this for Theorem 1.2.)

As in [6], the key is to prove that $i\partial\bar{\partial}\eta$ is invariant under translation by *arbitrary* flat sections of the Gauss-Manin connection on $R^1 f_* \mathbb{R} \otimes \mathcal{C}_B^\infty$. Concretely, we must show that if $T = (v_1, \dots, v_{2n})$,

then for all functions $\sigma : B \rightarrow \mathbb{C}^n$ of the form $\sigma(y) = \sum \lambda_i v_i(y)$ with constants $\lambda_i \in \mathbb{R}$, the difference $(\eta \circ T_\sigma) - \eta$ is pluriharmonic on $B \times \mathbb{C}^n$, where $T_\sigma(y, z) = (y, z + \sigma(y))$. Indeed, once we have this, then specializing to $\lambda_i \in \mathbb{Z}$ yields the desired invariance of $i\partial\bar{\partial}\eta$ under the deck group; moreover, it then suffices to check that $i\partial\bar{\partial}\eta \geq 0$ at the zero section $z = 0$, which is clear from (2.1) (the vertical components are given by ω , and the horizontal and mixed ones vanish; compare [6, p. 529]).

The proof of the required translation property is similar to the one in [6, p. 529]. Indeed, writing $S^{-1} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ with $A, B, C, D \in \mathbb{R}^{n \times n}$, a straightforward computation shows that

$$(\eta \circ T_\sigma) - \eta = \sum_{j,\ell=1}^n (\lambda_j C_{\ell j} + \lambda_{n+j} D_{\ell j}) (2\operatorname{Im} z_\ell + \sum_{k=1}^n (\lambda_k ((\operatorname{Im} Z)C)_{\ell k} + \lambda_{n+k} ((\operatorname{Im} Z)D)_{\ell k})),$$

which is obviously pluriharmonic. This completes the proof of Theorem 1.2.

Remark 2.5. In fact, we have proved a more precise result: given any Kähler form ω_M on M , there exists a semi-flat form $\omega_{\text{SF}} \geq 0$ on U as in Theorem 1.2 such that $\omega_{\text{SF}}|_{M_y}$ is cohomologous to $\omega_M|_{M_y}$ for all $y \in B$. But this is not required in any of the following. In particular, unlike in [6], there will be no need to construct ω_{SF} using the same ω_M that was used to construct $\tilde{\omega}_t$.

3. MAIN ESTIMATES

The proof of Theorem 1.1 given in [6], under the assumption that M is projective, has two parts: (a) the C^0 estimate of $\tilde{\omega}_t$, and (b) the C^k estimate of $\tilde{\omega}_t$ for $k \geq 1$. The projectivity of M was used in the course of the construction of ω_{SF} in (a), and in (b) through the proof of a certain $i\partial\bar{\partial}$ -lemma for abelian fibrations [6, Proposition 3.1] analogous to [7, Lemma 4.3] and [8, Proposition 3.7].

Thanks to Theorem 1.2, (a) now goes through verbatim without projectivity. Using the work in Section 2, the $i\partial\bar{\partial}$ -lemma could easily be extended to the general Kähler setting as well, and then the rest of (b) would go through without changes. However, we will give a simpler and more robust argument for (b) here by working at the level of metrics rather than potentials, employing a local version of Yau's C^3 estimate. This eliminates the rather difficult $i\partial\bar{\partial}$ -lemma from the proof.

Throughout this section, we will assume that the compact set K of Theorem 1.1 is so small that it can be identified with one of its preimages under the universal covering map $p : B \times \mathbb{C}^n \rightarrow U$ for some sufficiently small coordinate ball $B \subset N \setminus f(S)$, where $U = f^{-1}(B)$.

3.1. The C^0 estimate of $\tilde{\omega}_t$. Let $\omega_{\text{SF}} \geq 0$ be the semi-flat form on U constructed in Theorem 1.2, so that $\omega_0 + \omega_{\text{SF}}$ is a semi-flat Kähler form on U . As in [6] define $\lambda_t : B \times \mathbb{C}^n \rightarrow B \times \mathbb{C}^n$ by

$$\lambda_t(y, z) = \left(y, \frac{z}{\sqrt{t}} \right).$$

Given Theorem 1.2, the same proof as in [6, Lemma 4.2] (which relies on some estimates from [12]) shows that there exists a constant C_B such that

$$C_B^{-1} p^*(\omega_0 + \omega_{\text{SF}}) \leq \lambda_t^* p^* \tilde{\omega}_t \leq C_B p^*(\omega_0 + \omega_{\text{SF}})$$

holds on $B \times \mathbb{C}^n$, uniformly in $t \in (0, 1]$. This trivially implies that for each compact set $K \subset B \times \mathbb{C}^n$ there exists a constant C_K such that, with δ denoting the Euclidean metric on $B \times \mathbb{C}^n$,

$$C_K^{-1} \delta \leq \lambda_t^* p^* \tilde{\omega}_t \leq C_K \delta. \quad (3.1)$$

In fact, C_K only depends on $B \subset N \setminus f(S)$ if we take K to lie in a fixed fundamental domain of the deck group action, which we can. Note that (3.1) implies (1.1) for $k = 0$ because $t \leq 1$.

3.2. The C^1 estimate of $\tilde{\omega}_t$. The following is a simple modification of an argument of Sherman-Weinkove [11] for the Kähler-Ricci flow. For simplicity of notation, we denote by g the Riemannian metric associated with the Kähler form $\lambda_t^* p^* \tilde{\omega}_t$, but of course g still depends on t .

Proposition 3.1. *Given any compact set $K \subset B \times \mathbb{C}^n$, there exists a constant C_K such that*

$$\sup_K |\nabla g|^2 \leq C_K \quad (3.2)$$

holds uniformly for all $t \in (0, 1]$, where the connection and norm are the ones associated with δ .

Proof. Fix a slightly larger compact set K' containing K in its interior. On K' , by (3.1),

$$C^{-1}\delta \leq g \leq C\delta \quad (3.3)$$

for some generic constant $C = C_{K'} = C_K$ independent of t . Let $\psi \geq 0$ be a smooth cutoff function supported in K' with $\psi \equiv 1$ on K such that $|\nabla \psi|^2 \leq C$ and $\Delta(\psi^2) \geq -C$, where the gradient and Laplacian are the ones associated with δ . Then, by (3.3), $|\nabla_g \psi|_g^2 \leq C$ and $\Delta_g(\psi^2) \geq -C$.

Following Yau [14], we define

$$S = |\nabla g|_g^2 = g^{i\bar{\ell}} g^{j\bar{q}} g^{p\bar{k}} \partial_i g_{j\bar{k}} \partial_{\bar{\ell}} g_{p\bar{q}}.$$

The norm (but not the connection) is now the one associated with g , which makes no difference for the final estimate because of (3.3). If T denotes the difference of the Christoffel symbols of g and δ , restricted to the $(1, 0)$ -tangent bundle, then T is a tensor and it is easy to see that $S = |T|_g^2$.

Using that g is Ricci-flat and δ is flat, the version of the Calabi-Yau C^3 estimate in [9] gives

$$\Delta_g S = |\nabla_g T|_g^2 + |\bar{\nabla}_g T|_g^2.$$

Notice here that T is not a real-valued tensor. We can then compute that

$$\Delta_g(\psi^2 S) \geq \psi^2 (|\nabla_g T|_g^2 + |\bar{\nabla}_g T|_g^2) - CS - 2|\langle \nabla_g \psi^2, \nabla_g S \rangle_g| \geq -CS,$$

where we have used that, by Young's inequality,

$$2|\langle \nabla_g \psi^2, \nabla_g S \rangle_g| = 4\psi |\langle \nabla_g \psi, \nabla_g |T|_g^2 \rangle_g| \leq C\psi |\nabla_g |T|_g^2|_g \leq \psi^2 (|\nabla_g T|_g^2 + |\bar{\nabla}_g T|_g^2) + CS.$$

On the other hand, the Aubin-Yau C^2 estimate, using again that g is Ricci-flat and δ is flat, gives

$$\Delta_g \text{tr}_\delta g = \delta^{i\bar{\ell}} g^{j\bar{q}} g^{p\bar{k}} \partial_i g_{j\bar{k}} \partial_{\bar{\ell}} g_{p\bar{q}} \geq C^{-1} S,$$

using (3.3). Thus, if we pick C' large enough depending on the value of C up to here, then

$$\Delta_g(\psi^2 S + C' \text{tr}_\delta g) \geq 0.$$

Hence the maximum of $\psi^2 S + C' \text{tr}_\delta g$ in K' is achieved on the boundary of K' , which implies that $\sup_K S \leq \sup_{K'} (\psi^2 S + C' \text{tr}_\delta g) \leq C' \sup_{\partial K'} \text{tr}_\delta g \leq C$ as required, using (3.3). \square

Now (3.2) indeed implies (1.1) for $k = 1$, again because $t \leq 1$; compare [6, Lemma 4.5].

3.3. Higher order estimates. To prove (1.1) for $k \geq 2$, we use a standard bootstrap argument. Since g is Ricci-flat Kähler, we have that $\partial_i \partial_{\bar{j}} \log \det(g_{k\bar{\ell}}) = 0$ for all $i, j \in \{1, \dots, n\}$. This implies that the component functions of g satisfy the quasilinear elliptic system

$$\Delta_g(g_{i\bar{j}}) = Q_{i\bar{j}} = \sum g^{k\bar{q}} g^{p\bar{\ell}} \partial_i g_{k\bar{\ell}} \partial_{\bar{j}} g_{p\bar{q}}. \quad (3.4)$$

For the bootstrap we also require three suitably nested compact regions $K'' \supset K' \supset K$.

By (3.1) and (3.2), $\|Q_{i\bar{j}}\|_{L^p(K'', \delta)} \leq C_K$ for all $p \geq 1$. Thus, $\|g_{i\bar{j}}\|_{W^{2,p}(K', \delta)} \leq C_{K,p}$ for $p > 1$ by L^p regularity theory since the coefficient matrix of (3.4) has bounded ellipticity and bounded modulus of continuity by (3.1), (3.2). Then $\|g_{i\bar{j}}\|_{C^{1,\alpha}(K', \delta)} \leq C_{K,\alpha}$ for all $\alpha \in (0, 1)$ by Morrey's inequality.

Now whenever g is bounded in $C^{k,\alpha}(K'', \delta)$ for some $k \geq 1$, then Q is bounded in $C^{k-1,\alpha}(K'', \delta)$, so that g is bounded in $C^{k+1,\alpha}(K', \delta)$ by interior Schauder theory, which can be used here because the coefficients of (3.4) are trivially bounded in $C^{k-1,\alpha}(K'', \delta)$ by assumption. Thus, shrinking and relabeling K'' and K' in each step, we inductively prove that $\|g\|_{C^k(K, \delta)} \leq C_{K,k}$ for all $k \geq 2$. Again these estimates imply (1.1) for the corresponding values of k because $t \leq 1$.

This completes the proof of Theorem 1.1.

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